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On time-like surfaces of positive constant Gaussian curvature and imaginary principal curvatures

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Abstract

We establish the Bäcklund transformation for the construction of time-like surfaces with positive Gaussian curvature and imaginary principal curvatures. The construction can be realized by algebraic algorithm via Darboux transformations. © 2002 Elsevier Science B.V. All rights reserved.

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0. Introduction

In 19th century, the construction of surfaces of negative constant curvature in Euclidean space $\boldsymbol{R}^3$ was one of the most important problems in differential geometry [7,26]. The related topics, such as sinh-Gordon equation, Bäcklund transformations [1] and Darboux transformations [5]have been developed extensively in the second half of 20th century and constitute an essential part in the modern soliton theory [10,19,23,26]. In theory of relativity, geometry of indefinite metric is very crucial. Hence, the theory of surfaces in Minkowski space $\boldsymbol{R}^{2,1}$ which has the metric $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}z^2$ attracted much attention. A series of papers are devoted to the construction of surfaces of constant mean curvature [6,14,17,18] or constant Gaussian curvature [4,12–15,20,22,25]. The situation is much more complicated than the Euclidean case, since the surfaces may have a definite metric (space-like surfaces), Lorentz metric (time-like surfaces) or mixed metric.

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Recently, the time-like surface of constant mean curvature in $\boldsymbol{R}^{2,1}$ have been studied systematically [6,14,17,18]. In the meantime, the four kinds of surfaces (space-like and time-like surfaces of positive and negative constant curvature) have been considered in [11]by a unified approach—*Darboux transformation*. It should be mentioned that a time-like surface of positive constant Gaussian curvature is a parallel surface of a time-like surface of constant mean curvature. In [18], it is pointed that there is a class of surfaces of constant mean curvature whose principal curvatures are imaginary (see also [21]).

This class of surfaces has not been studied in [18] and has been missed in [11]. In [16,20], the time-like pseudo-spherical congruences whose two focal surfaces are time-like and of positive constant Gaussian curvature have been considered. However, the case of imaginary principal curvatures has not been studied either. The purpose of the present paper is to study the class of time-like surfaces with positive constant Gaussian curvature and imaginary principal curvatures. It is seen that these surfaces have real asymptotic lines, and hence, we can used some special asymptotic coordinates. The surfaces can be determined by a solution of cosh-Gordon equation

$$\omega_{uv} = \cosh\omega.$$

By using the time-like and space-like pseudo-spherical congruences the Bäcklund transformation approach for the cosh-Gordon equation is established. Moreover, it is proved that the system of partial differential equations can be solved explicitly by using the Darboux transformation. The construction can be continued successively via some algebraic algorithm.

We add a list of surfaces of constant Gaussian curvature and pseudo-spherical congruences in the Appendix A.

Moreover, we would like to mention another motivation (from theoretical physics) to the study of time-like surfaces of constant positive curvature with imaginary principal curvatures.

The relativistic string is the one-dimensional relativistic object whose time evolution extrimizes the Nambu-Goto action

$$\mathcal{S}(\boldsymbol{r}) = -\kappa \int \mathrm{d}A,$$

where dA is the area element of the world sheet and κ is a constant.

From the mathematical point of view, world sheet of the relativistic string is a *time-like minimal* surface in the spacetime (cf. [2,8]).

In [3], Barbashov et al. suggested a generalization of relativistic string model in three-dimensional spacetimes with the action

$$\mathcal{S}_H(\boldsymbol{r}) := \mathcal{S}(\boldsymbol{r}) - 2\kappa H \int \mathrm{d}V$$

called *relativistic string with an external field*. Here, dV is the volume element of the region bounded by the world sheet and H is a nonzero constant.

One can easily to see that critical points of this action integral are time-like surfaces with *nonzero* constant mean curvature H in the spacetime.

Moreover, they mentioned a relation between this generalized string model and soliton theory. More precisely they claimed that the field equation of this generalized model in three-dimensional Minkowski space $\boldsymbol{R}^{2,1}$ coincides with the following sinh-Gordon equation:

$$\omega_{uv} = \sinh \omega.$$

In their argument, they took isothermal-curvature line coordinates for the world sheet. Namely, they assumed implicitly that the world sheet has two real distinct principal curvatures everywhere.

However, as we explained above, time-like surfaces of constant mean curvature may have imaginary real principal curvatures.

This observation also motivates us to study the soliton theory and differential geometry of cosh-Gordon equation and the corresponding time-like surfaces.

1. Asymptotic Chebyshev coordinates and cosh-Gordon equation

Let $\boldsymbol{R}^{2,1}$ be Minkowski three-space with Lorentzian metric $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}z^2$, and S be a connected orientable two-manifold and $\boldsymbol{r} : S \to \boldsymbol{R}^{2,1}$ an immersion. The immersion $\boldsymbol{r}$ is said to be *time-like* if the induced metric $\mathrm{I} := \mathrm{d}\boldsymbol{r} \cdot \mathrm{d}\boldsymbol{r}$ of S is Lorentzian. We use the abbreviation "·" for the scalar product of vectors in Lorentzian metric $\mathrm{d}s^2$.

The unit normal vector field $\boldsymbol{n}$ of S can be regarded as a smooth map $\boldsymbol{n} : S \to \boldsymbol{S}^{1,1}$ into the unit pseudosphere

$$\boldsymbol{S}^{1,1} := \{(x, y, z) \in \boldsymbol{R}^{2,1} | x^2 + y^2 - z^2 = 1\}$$

and called *Gauss map* of S.

Let S be a time-like surface in $\boldsymbol{R}^{2,1}$ with unit normal vector field $\boldsymbol{n}$. We can introduce a system of frames $\{\boldsymbol{r}; \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{n}\}$ such that $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are tangential null vector fields. Thus,

$$\boldsymbol{e}_a^2 = 0, \qquad \boldsymbol{n}^2 = 1, \qquad \boldsymbol{n} \cdot \boldsymbol{e}_a = 0, \quad (a = 1, 2). \tag{1.1}$$

The fundamental equations of the surface are

$$\mathrm{d}\boldsymbol{r} = \omega^a \boldsymbol{e}_a, \qquad \mathrm{d}\boldsymbol{e}_a = \omega_a^b \boldsymbol{e}_b + \omega_a^3 \boldsymbol{n}, \qquad \mathrm{d}\boldsymbol{n} = \omega_3^a \boldsymbol{e}_a, \quad (a = 1, 2). \tag{1.2}$$

From Eq. (1.1) it follows

$$\omega_1^2 = \omega_2^1 = 0, \tag{1.3}$$

and

$$\omega_1^3 + \tfrac{1}{2}\mathrm{e}^{\omega}\omega_3^2 = 0, \qquad \omega_2^3 + \tfrac{1}{2}\mathrm{e}^{\omega}\omega_3^1 = 0. \tag{1.4}$$

Here, we assume $\boldsymbol{e}_1 \cdot \boldsymbol{e}_2$ to be positive and equal to $\mathrm{e}^{\omega}/2$. We suppose that there exist special coordinates (u, v) such that

$$\omega^1 = \mathrm{d}u - \mathrm{e}^{-\omega}\,\mathrm{d}v, \qquad \omega^2 = -\mathrm{e}^{-\omega}\,\mathrm{d}u - \mathrm{d}v \tag{1.5}$$

and

$$\omega_3^1 = -\mathrm{d}u - \mathrm{e}^{-\omega}\,\mathrm{d}v, \qquad \omega_3^2 = -\mathrm{e}^{-\omega}\,\mathrm{d}u + \mathrm{d}v. \tag{1.6}$$

Afterward we will see that this is equivalent to say that the surface is of $K = 1$ and the principal curvatures are imaginary. From Eq. (1.4), we have

$$\omega_1^3 = \tfrac{1}{2}(\mathrm{d}u - \mathrm{e}^{\omega}\,\mathrm{d}v), \qquad \omega_2^3 = \tfrac{1}{2}(\mathrm{e}^{\omega}\,\mathrm{d}u + \mathrm{d}v)$$

From $\mathrm{d}^2\boldsymbol{r} = 0$, we obtain

$$\omega_1^1 = \omega_u\,\mathrm{d}u, \qquad \omega_2^2 = \omega_v\,\mathrm{d}v. \tag{1.7}$$

The first and second fundamental forms are, respectively

$$\mathrm{I} = \mathrm{d}\boldsymbol{r}^2 = -\mathrm{d}u^2 - 2\sinh\omega\,\mathrm{d}u\,\mathrm{d}v + \mathrm{d}v^2, \tag{1.8}$$

$$\mathrm{II} = -\mathrm{d}\boldsymbol{n}\cdot\mathrm{d}\boldsymbol{r} = -2\cosh\omega\,\mathrm{d}u\,\mathrm{d}v \tag{1.9}$$

It is easily seen that $K = 1$ and the principal curvatures of S are imaginary. The *Gauss equation* $\mathrm{d}\omega_b^a + \omega_c^a \wedge \omega_b^c = -\omega_3^a \wedge \omega_b^3$ is

$$\omega_{uv} = \cosh\omega. \tag{1.10}$$

It is seen that *Codazzi equations*

$$\mathrm{d}\omega_3^1 + \omega_1^1 \wedge \omega_3^1 = 0, \qquad \mathrm{d}\omega_3^2 + \omega_2^2 \wedge \omega_3^2 = 0 \tag{1.11}$$

or

$$\mathrm{d}\omega_1^3 + \omega_1^3 \wedge \omega_1^1 = 0, \qquad \mathrm{d}\omega_2^3 + \omega_2^3 \wedge \omega_2^2 = 0 \tag{1.12}$$

hold true. Thus, from the fundamental theorem of surfaces, we have

Theorem 1.1. *From a solution ω of the cosh-Gordon* Eq. (1.10) *there is a time-like surface S with $K = 1$ and imaginary principal curvatures.*

According to the expressions (1.8) and (1.9) of the fundamental forms I and II, we call the coordinates (u, v) *asymptotic Chebyshev coordinates*.

Remark 1.2. Similar as that in Euclidean case, the surfaces S of $K = 1$ in $\boldsymbol{R}^{2,1}$ is a parallel surface of a surface of constant mean curvature $H = 1/2$ with distance 1. From the conformal flat coordinates of time-like surface of constant mean curvature [17], we see the existence of asymptotic Chebyshev coordinates of S.

Remark 1.3. The cosh-Gordon equation has been appeared in the study of the Cauchy problem of the harmonic maps from $\boldsymbol{R}^{1,1}$ into $\boldsymbol{S}^{1,1}$ [9].

2. Bäcklund transformations

Let S be a known time-like surface with $K = 1$ and two imaginary principal curvatures (u, v) be the asymptotic Chebyshev coordinates. We use the pseudo-spherical congruences to construct surfaces of the same characters. The congruences can be time-like and space-like either. Let

$$\boldsymbol{r}' = \boldsymbol{r} + l(a\boldsymbol{e}_1 + b\boldsymbol{e}_2). \tag{2.1}$$

Here l is a real constant and $ab\,\mathrm{e}^{\omega} = -1$. The lines $\boldsymbol{rr}'$ generate a *time-like line congruence* Σ_- and for the time-like vector $\overrightarrow{\boldsymbol{rr}'}$

$$(\overrightarrow{\boldsymbol{rr}'})^2 = -l^2. \tag{2.2}$$

Let S' be the surface generated by $\boldsymbol{r}'$. The surface S is a focal surface of Σ_-. We demand that S' is another focal surface of the congruence Σ_- and the unit normal vector field of S' should be

$$\boldsymbol{n}' = \cos\tau(a\boldsymbol{e}_1 - b\boldsymbol{e}_2) + \sin\tau\boldsymbol{n} \tag{2.3}$$

which is evidently perpendicular to the line $\overline{\boldsymbol{rr}'}$. Here, $\sin\tau = \boldsymbol{n}\cdot\boldsymbol{n}'$ is a constant. Thus, Σ_- is a time-like pseudo-spherical congruence in $\boldsymbol{R}^{2,1}$. In order that S' is another focal surface of Σ_-, we should have $\boldsymbol{n}'\cdot \mathrm{d}\boldsymbol{r}' = 0$. From

$$\mathrm{d}\boldsymbol{r}' = \mathrm{d}\boldsymbol{r} + l(\mathrm{d}a\boldsymbol{e}_1 + a\mathrm{d}\boldsymbol{e}_1 + \mathrm{d}b\boldsymbol{e}_2 + b\mathrm{d}\boldsymbol{e}_2) \tag{2.4}$$

and the fundamental equations, we have

$$\begin{aligned}\mathrm{d}\boldsymbol{r}' = {}&[(\mathrm{d}u - \mathrm{e}^{-\omega}\,\mathrm{d}v) + l(a_u\,\mathrm{d}u + a_v\,\mathrm{d}v) + la\omega_u\,\mathrm{d}u]\boldsymbol{e}_1\\ &+[(-\mathrm{e}^{-\omega}\,\mathrm{d}u - \mathrm{d}v) + l(b_u\,\mathrm{d}u + b_v\,\mathrm{d}v) + lb\omega_v\,\mathrm{d}v]\boldsymbol{e}_2\\ &+\frac{l}{2}[a(\mathrm{d}u - \mathrm{e}^{\omega}\,\mathrm{d}v) + b(\mathrm{e}^{\omega}\,\mathrm{d}u + \mathrm{d}v)]\boldsymbol{n},\end{aligned} \tag{2.5}$$

or

$$\begin{aligned}\boldsymbol{r}'_u &= (1 + la_u + la\omega_u)\boldsymbol{e}_1 + (-\mathrm{e}^{-\omega} + lb_u)\boldsymbol{e}_2 + \frac{l}{2}(a + b\,\mathrm{e}^{\omega})\boldsymbol{n},\\ \boldsymbol{r}'_v &= (-\mathrm{e}^{-\omega} + la_v)\boldsymbol{e}_1 + (-1 + lb_v + lb\omega_v)\boldsymbol{e}_2 + \frac{l}{2}(-a\,\mathrm{e}^{\omega} + b)\boldsymbol{n}.\end{aligned} \tag{2.6}$$

By using $ab\,\mathrm{e}^{\omega} = -1$ and

$$ab_u + a_ub + ab\omega_u = 0, \qquad ab_v + a_vb + ab\omega_v = 0 \tag{2.7}$$

we can write

$$\begin{aligned}\boldsymbol{r}'_u &= \left(1 - la\frac{b_u}{b}\right)\boldsymbol{e}_1 + (-\mathrm{e}^{-\omega} + lb_u)\boldsymbol{e}_2 + \frac{l}{2}(a - a^{-1})\boldsymbol{n},\\ \boldsymbol{r}'_v &= (-\mathrm{e}^{-\omega} + la_v)\boldsymbol{e}_1 + \left(-1 - lb\frac{a_v}{a}\right)\boldsymbol{e}_2 + \frac{l}{2}(b + b^{-1})\boldsymbol{n}.\end{aligned} \tag{2.8}$$

Take $l = \cos\tau$ and define $\mu = \sec\tau - \mathrm{tg}\tau$, the equations $\boldsymbol{n}'\cdot\mathrm{d}\boldsymbol{r}' = 0$ can be written as

$$2b^{-1}b_u = -\mu(a - a^{-1}), \qquad 2a^{-1}a_v = -\frac{1}{\mu}(b + b^{-1}) \tag{2.9}$$

Without loss of generalities, we may assume that a is positive and put

$$a := \exp\frac{\omega' - \omega}{2}, \qquad b := -\exp\frac{-\omega' - \omega}{2}. \tag{2.10}$$

Then, we obtain the following system of equations:

$$(\omega' + \omega)_u = 2\mu \sinh \frac{\omega' - \omega}{2}, \qquad (\omega' - \omega)_v = \frac{2}{\mu} \cosh \frac{\omega' + \omega}{2}. \tag{2.11}$$

By differentiation, it is seen that

$$\begin{aligned}(\omega' + \omega)_{uv} &= 2 \cosh \frac{\omega' - \omega}{2} \cosh \frac{\omega' + \omega}{2},\\ (\omega' - \omega)_{uv} &= 2 \sinh \frac{\omega' - \omega}{2} \sinh \frac{\omega' + \omega}{2}.\end{aligned} \tag{2.12}$$

The integrability condition of (2.11) for ω' is just the cosh-Gordon equation (1.10) and the solution ω' satisfies (1.10) too. Thus, Eq. (2.11) describes the *Bäcklund transformation* $\omega \mapsto \omega'$ for the cosh-Gordon equation. We may use the Bäcklund's theorem to confirm that S' is a time-like surface with $K = 1$. However, for the purpose of applying the Bäcklund transformation successively and showing that S' has two imaginary principal curvatures, we have to prove that (u, v) are also the asymptotic Chebyshev coordinates of S'. By direct but long calculation we obtain

$$\boldsymbol{r}_u'^2 = -1, \qquad \boldsymbol{r}_v'^2 = 1, \qquad \boldsymbol{r}_u' \cdot \boldsymbol{r}_v' = -\sinh \omega'. \tag{2.13}$$

Hence, the first fundamental form of S' is

$$\mathrm{I}' = -\mathrm{d}u^2 - 2 \sinh \omega' \,\mathrm{d}u\, \mathrm{d}v + \mathrm{d}v^2. \tag{2.14}$$

Moreover, by differenting (2.3), we obtain

$$\begin{aligned}\boldsymbol{n}_u' &= \left(-a\frac{b_u}{b} \cos\tau - \sin\tau\right) \boldsymbol{e}_1 + (-b_u \cos\tau - \mathrm{e}^{-\omega} \sin\tau)\boldsymbol{e}_2 + \frac{\cos\tau}{2}(a + a^{-1})\boldsymbol{n},\\ \boldsymbol{n}_v' &= (a_v \cos\tau - \mathrm{e}^{-\omega} \sin\tau)\boldsymbol{e}_1 + \left(b\frac{a_v}{a} + \sin\tau\right) \boldsymbol{e}_2 - \frac{\cos\tau}{2}(b - b^{-1})\boldsymbol{n}.\end{aligned} \tag{2.15}$$

By calculation, we have

$$-\boldsymbol{n}_u \cdot \boldsymbol{r}_u = 0, \qquad -\boldsymbol{n}_v \cdot \boldsymbol{r}_v = 0, \qquad -(\boldsymbol{n}_u \cdot \boldsymbol{r}_v + \boldsymbol{n}_v \cdot \boldsymbol{r}_u) = -2 \cosh \omega'. \tag{2.16}$$

Thus, the second fundamental for S' is

$$\mathrm{II}' = -2 \cosh \omega' \,\mathrm{d}u\, \mathrm{d}v. \tag{2.17}$$

From Eqs. (2.15) and (2.17) it is seen that S' is a surface of $K = 1$ and has two imaginary principal curvatures. Moreover, (u, v) are the asymptotic Chebyshev coordinates of S'. Consequently, the Bäcklund transformation can be done successively, if the system (2.12) can be solved successively.

The above results can be summarized as the following theorem:

Theorem 2.1 (Time-like Bäcklund transformation). *Let S be a time-like surface with $K=1$ and two imaginary principal curvature and (u,v) be the asymptotic Chebyshev coordinates, then*

1. *the system of* Eq. (2.11) *is completely integrable*;
2. *if ω' is a solution of* (2.11), *then* Eq. (2.1) *in which, a, b have the expression* (2.10) *defines a time-like surface S' with $K=1$ and two imaginary principal curvatures*;
3. *the coordinates (u,v) are asymptotic Chebyshev coordinates of S' too.*

Now, we use a space-like congruence to construct S' from S. We choose $ab\,\mathrm{e}^{\omega}=1$ in Eq. (2.1), then a *space-like congruence* Σ_+ is obtained. In this case, we should have

$$\boldsymbol{n}' = \sinh\tau\,(a\boldsymbol{e}_1 - b\boldsymbol{e}_2) + \cosh\tau\,\boldsymbol{n} \tag{2.18}$$

which is still a unit space-like vector field. The Eq. (2.6) holds as well and (2.8) is replaced by

$$\begin{aligned}\boldsymbol{r}'_u &= \left(1 - la\frac{b_u}{b}\right)\boldsymbol{e}_1 + (-\mathrm{e}^{-\omega} + lb_u)\boldsymbol{e}_2 + \frac{l}{2}(a+a^{-1})\boldsymbol{n},\\ \boldsymbol{r}'_v &= (-\mathrm{e}^{-\omega} + la_v)\boldsymbol{e}_1 + \left(-1 - lb\frac{a_v}{a}\right)\boldsymbol{e}_2 - \frac{l}{2}(b^{-1}-b)\boldsymbol{n}.\end{aligned} \tag{2.19}$$

Let $l=\sinh\tau$, the condition $\boldsymbol{n}'\cdot \mathrm{d}\boldsymbol{r}'=0$ becomes

$$\frac{2b_u}{b} = \mu(a+a^{-1}), \qquad \frac{2a_v}{a} = -\frac{1}{\mu}(b-b^{-1}). \tag{2.20}$$

Here, $\mu = \mathrm{csch}\tau - \mathrm{cth}\tau$. Let

$$a = \exp\frac{-\omega'-\omega}{2}, \qquad b = \exp\frac{\omega'-\omega}{2}. \tag{2.21}$$

Then, we obtain

$$(\omega'-\omega)_u = 2\mu\cosh\frac{\omega'+\omega}{2}, \qquad (\omega'+\omega)_v = \frac{2}{\mu}\sinh\frac{\omega'-\omega}{2}. \tag{2.22}$$

This is the equations for Bäcklund transformation of the cosh-Gordon equations too. It is seen that (2.22) becomes (2.11), if we interchange (u,v) and use $(1/\mu)$ instead of μ. Further calculations imply the anologue of Theorem 2.1.

Theorem 2.2 (Space-like Bäcklund transformation). *Let S be a time-like surface with $K=1$ and two imaginary principal curvatures and (u,v) be the asymptotic Chebyshev coordinates, then*

1. *the system* (2.22) *is completely integrable*;
2. *if ω' is a solution of* (2.22), *then* (2.1), *in which, a, b have the expressions* (2.21), *defines a time-like surface S' with $K=1$ and two imaginary principal curvatures*;
3. *the coordinates (u,v) are the asymptotic Chebyshev coordinates of S'* too.

3. Darboux transformations

We use Darboux transformations to realize the Bäcklund transformations explicitly. At first, we have

Lemma 3.1. *The cosh-Gordon equation is the zero curvature condition of the Lax pair*

$$U := \Phi_u\Phi^{-1} = \frac{\lambda}{2}\begin{bmatrix} 0 & -e^{-\omega} \\ e^{\omega} & 0 \end{bmatrix}, \qquad V := \Phi_v\Phi^{-1} = \frac{1}{2}\begin{bmatrix} -\omega_v & \dfrac{1}{\lambda} \\ \dfrac{1}{\lambda} & \omega_v \end{bmatrix}. \tag{3.1}$$

Here λ ($\neq 0$) is the spectral parameter and Φ is a matrix valued function, depending on the real variable(u, v) and the complex variable $\lambda(\lambda \neq 0)$.

Proof. By direct calculation, we see that the zero curvature condition

$$U_v - V_u + [U, V] = 0 \tag{3.2}$$

is equivalent to the cosh-Gordon equation. □

Lemma 3.2. *If $(h_1, h_2)^t$ is a column solution to the Lax pair for $\lambda = \lambda_0$ ($h_1 \neq 0, h_2 \neq 0$), then $(-h_1, h_2)^t$ is a column solution to the Lax pair for $\lambda = -\lambda_0$.*

Proof. Substitute $(-h_1, h_2)^t$ in the Lax pair (3.1) for $\lambda = -\lambda_0$, it is easily seen that the Lax pair is satisfied. □

Lemma 3.3. *Let ω be a real solution of cosh-Gordon equation, λ_0 pure-imaginary number and $(h_1, h_2)^t$ a column solution of the Lax pair for $\lambda = \lambda_0$. If h_2/h_1 is pure-imaginary at one point (u_0, v_0), then it is pure-imaginary on any connected region containing (u_0, v_0), where the solution $(h_1, h_2)^t$ makes sense and $h_1 \neq 0$, $h_2 \neq 0$.*

Proof. From the Lax equation (3.1), we see that

$$\left(\frac{h_2}{h_1}\right)_u = \frac{\lambda_0}{2}\left[e^{\omega} + e^{-\omega}\left(\frac{h_2}{h_1}\right)^2\right], \tag{3.3}$$

$$\left(\frac{h_2}{h_1}\right)_v = \frac{1}{2\lambda_0} + \omega_v\left(\frac{h_2}{h_1}\right) - \frac{1}{2\lambda_0}\left(\frac{h_2}{h_1}\right)^2. \tag{3.4}$$

Let $A = h_2/h_1 + \overline{h_2/h_1}$. Then

$$A_u = \frac{\bar{\lambda}_0}{2}e^{-\omega}\left(\frac{\bar{h}_2}{\bar{h}_1} - \frac{h_2}{h_1}\right)A, \qquad A_v = \omega_v A + \frac{1}{2\bar{\lambda}_0}\left(\frac{h_2}{h_1} - \frac{\bar{h}_2}{\bar{h}_1}\right)A. \tag{3.5}$$

Consequently, if $A = 0$ at (u_0, v_0), then $A = 0$ on any connected region containing (u_0, v_0). Hence, h_2/h_1 is pure-imaginary. □

Theorem 3.4 (Darboux transformation). *If ω is a real solution of cosh-Gordon equation, then the function ω', defined by*

$$e^{\omega'} := -\left(\frac{h_2}{h_1}\right)^2 e^{-\omega} \tag{3.6}$$

is a real solution of the cosh-Gordon equation too. Here, $(h_1, h_2)^t$ is a column solution of the Lax Eq. (3.1) *corresponding to a pure-imaginary spectral parameter λ_0 and h_2/h_1 is pure-imaginary.*

Proof. First, we notice that the preceding Lemma implies that it is possible to have pure-imaginary h_2/h_1.

From the general theory of Darboux transformation in matrix form [10,11]

$$\Phi^1 = \mathcal{D}(\lambda)\,\Phi, \qquad \mathcal{D}(\lambda) = I + \lambda\mathcal{A} \tag{3.7}$$

is a solution of the Lax equation (3.1)for some ω'. Here,

$$\mathcal{A} = -H\begin{bmatrix} \frac{1}{\lambda_0} & 0 \\ 0 & -\frac{1}{\lambda_0} \end{bmatrix} H^{-1} = -\frac{1}{\lambda_0}\begin{bmatrix} 0 & \frac{h_1}{h_2} \\ \frac{h_2}{h_1} & 0 \end{bmatrix} \tag{3.8}$$

with $H = \begin{bmatrix} h_1 & -h_1 \\ h_2 & h_2 \end{bmatrix}$. Hence,

$$\Phi^1 = \left(I - \frac{\lambda}{\lambda_0}\begin{bmatrix} 0 & \frac{h_1}{h_2} \\ \frac{h_2}{h_1} & 0 \end{bmatrix}\right)\Phi. \tag{3.9}$$

We should have

$$\Phi^1_u = \frac{\lambda}{2}\begin{bmatrix} 0 & -e^{-\omega'} \\ e^{\omega'} & 0 \end{bmatrix}\Phi^1, \qquad \Phi^1_v = \frac{1}{2}\begin{bmatrix} -\omega'_v & \frac{1}{\lambda} \\ \frac{1}{\lambda} & \omega'_v \end{bmatrix}\Phi^1. \tag{3.10}$$

Substituting (3.9) into (3.10), we obtain (3.6). Besides, the zero-curvature condition of (3.10) implies that ω' is a solution of cosh-Gordon equation. □

The transformation $(\omega, \Phi) \mapsto (\omega', \Phi^1)$ is called the *Darboux transformation* (DT) and $\mathcal{D}$ the *Darboux matrix*. The new solution (ω', Φ^1) is called the *Darboux transform* of (ω, Φ).

It is noted that the conclusion of Theorem 3.4 can be deduced by somewhat tedious and lengthy but straightforward calculations. However, the explicit formula (3.7) of the matrix-function Φ^1 is necessary for applying DT successively.

Theorem 3.5. *The solution ω', obtained by the Darboux transformation, and the seed solution ω to the cosh-Gordon equation are related by the Bäcklund transformation* (2.11).

Proof. From Eq. (3.6), we see that

$$e^{\omega'+\omega} = -\left(\frac{h_2}{h_1}\right)^2. \tag{3.11}$$

Differentiating (3.11) and using (3.3), we obtain

$$(\omega' + \omega)_u = 2\mu \sinh \frac{\omega' - \omega}{2} \tag{3.12}$$

Here μ is a nonzero real number defined by $\mu = -\sigma\sqrt{-1}\lambda_0 \in \boldsymbol{R}^* = \boldsymbol{R} \setminus \{0\}$, where $\sigma = \pm 1$ such that

$$\exp\left\{\frac{\omega' + \omega}{2}\right\} = \sigma\sqrt{-1}\frac{h_2}{h_1} > 0.$$

Similarly, we have

$$(\omega' - \omega)_v = \frac{2}{\mu} \cosh(\omega' + \omega). \tag{3.13}$$

Consequently, Eq. (2.11) holds. This completes the proof. □

Theorem 3.5 implies the following result:

Corollary 3.6. *The (time-like) Bäcklund transformation* $\omega \mapsto \omega'$ *of cosh-Gordon equation can be solved explicitly by* Eq. (3.6), *provided a general solution* Φ *of the Lax equation* (3.1) *is known.*

Remark 3.7. The time-like surface of $K = 1$ with principal curvature $\kappa_1 = \kappa_2$ and free of umblics can be construct through Bäcklund transformation and Darboux transformation as well.

We sketch the procedure. Take

$$\omega^1 = \mathrm{d}u - e^{-\omega}\,\mathrm{d}v, \qquad \omega^2 = -\mathrm{d}v. \tag{3.14}$$

$$\omega_3^1 = -\mathrm{d}u - e^{-\omega}\,\mathrm{d}v, \qquad \omega_3^2 = \mathrm{d}v. \tag{3.15}$$

instead of Eqs. (1.4) and (1.6). We still take $\boldsymbol{e}_1^2 = \boldsymbol{e}_2^2 = 0$ and $\boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = e^{\omega}/2$. Then

$$\mathrm{I} = -e^{\omega}\,\mathrm{d}u\,\mathrm{d}v + \mathrm{d}v^2, \qquad \mathrm{II} = -e^{\omega}\,\mathrm{d}u\,\mathrm{d}v. \tag{3.16}$$

The Gauss equation becomes the Liouville equation

$$\omega_{uv} = \tfrac{1}{2}e^{\omega}. \tag{3.17}$$

The time-like Bäcklund transformation takes the form

$$\boldsymbol{r}' = \boldsymbol{r} + l(a\boldsymbol{e}_1 + b\boldsymbol{e}_2), \qquad ab\,e^{\omega} = -1, \qquad \boldsymbol{n}' = \cos\tau(a\boldsymbol{e}_1 - b\boldsymbol{e}_2) + \sin\tau\,\boldsymbol{n}. \tag{3.18}$$

Eqs. (2.11) become

$$(\omega' + \omega)_u = -\mu \exp\frac{\omega - \omega'}{2}, \qquad (\omega' - \omega)_v = \frac{2}{\mu}\cosh\frac{\omega + \omega'}{2}. \tag{3.19}$$

The integrability condition for ω' is the Liouville equation (3.17). By using the Lax pair

$$\Phi_u = \frac{\lambda}{2}\begin{bmatrix} 0 & 0 \\ e^{\omega} & 0 \end{bmatrix}\Phi, \qquad \Phi_v = \frac{1}{2}\begin{bmatrix} -\omega_v & \frac{1}{\lambda} \\ \frac{1}{\lambda} & \omega_v \end{bmatrix}\Phi, \tag{3.20}$$

we can apply Darboux transformation to construct explicit solutions. The space-like Bäcklund transformation takes the form

$$\boldsymbol{r}' = \boldsymbol{r} + l(a\boldsymbol{e}_1 + b\boldsymbol{e}_2), \qquad ab\,e^{\omega} = 1, \qquad \boldsymbol{n}' = \sinh\tau(a\boldsymbol{e}_1 - b\boldsymbol{e}_2) + \cosh\tau\boldsymbol{n}, \tag{3.21}$$

which can be treated in the same way.

4. Geometrical meaning of Lax pair

In this section, we give the geometrical meaning of the Lax pair (3.1) together with a *Sym formula* for time-like $K = 1$ surfaces with imaginary principal curvatures. To this end, we identify Minkowski three-space $\boldsymbol{R}^{2,1}$ with the Lie algebra $\underline{g} = \underline{sl}_2\boldsymbol{R}$ (cf. [17]).

We take the following basis $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ of $\underline{g}$:

$$\underline{e}_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \underline{e}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \underline{e}_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{4.1}$$

Hereafter, we identify $\boldsymbol{R}^{2,1}$ with $\underline{g}$ via this basis

$$(x, y, z) \leftrightarrow x\underline{e}_1 + y\underline{e}_2 + z\underline{e}_3. \tag{4.2}$$

By the linear isomorphism (4.2) the Lorentzian metric ds^2 corresponds to the scalar product

$$\langle X, Y\rangle = \tfrac{1}{2}\mathrm{tr}(XY), \quad X, Y \in \underline{g}. \tag{4.3}$$

The special linear group $G = SL_2\boldsymbol{R}$ acts isometrically on $\underline{g}$ via the Ad*-action $\mathrm{Ad}^* : \underline{g} \times G \to \underline{g}$

$$\mathrm{Ad}^*(\sigma)X = \mathrm{Ad}(\sigma^{-1})X = \sigma^{-1}X\sigma \quad, (\sigma \in G). \tag{4.4}$$

The Ad*-action induces a double covering $G \to SO^+(2, 1)$. Here, $SO^+(2, 1)$ denotes the identity component of the Lorentz group $O(2, 1)$.

The Ad*-action of G on $\boldsymbol{S}^{1,1}$ is transitive and isometric. The pseudosphere $\boldsymbol{S}^{1,1}$ is represented as

$$\boldsymbol{S}^{1,1} = \mathrm{Ad}^*(G)\underline{e}_1 = \{\mathrm{Ad}(\sigma^{-1})\underline{e}_1 | \sigma \in G\}. \tag{4.5}$$

Compare with [17], we have

Proposition 4.1. *Let $\Phi : D \times \boldsymbol{R}^* \to SL_2\boldsymbol{R}$ be a solution to the Lax pair* (3.1) *over a simply connected region D of S and $\lambda \in \boldsymbol{R}^*$. Then*

$$\boldsymbol{r}_\lambda := 2\Phi^{-1}\frac{\partial}{\partial t}\Phi, \quad \lambda = \pm \mathrm{e}^{-t} \tag{4.6}$$

describes a real loop of time-like $K = 1$ surfaces in $\boldsymbol{R}^{2,1}$ with imaginary principal curvatures. The unit normal vector field of each $\boldsymbol{r}_\lambda$ is given by

$$\boldsymbol{n}_\lambda = \mathrm{Ad}(\Phi^{-1})\underline{e}_1. \tag{4.7}$$

The first and second fundamental forms of each $\boldsymbol{r}_\lambda$ are given by

$$\mathrm{I}_\lambda = -\lambda^2\,\mathrm{d}u^2 - 2\sinh\omega\,\mathrm{d}u\,\mathrm{d}v + \lambda^{-2}\,\mathrm{d}v^2, \qquad \mathrm{II}_\lambda = -2\cosh\omega\,\mathrm{d}u\,\mathrm{d}v. \tag{4.8}$$

Proof. Under the identification (4.2), differentiating (4.6), we have

$$\frac{\partial}{\partial u}\boldsymbol{r}_\lambda = -\lambda\mathrm{Ad}(\Phi^{-1})\{\sinh\omega\,\underline{e}_2 + \cosh\omega\,\underline{e}_3\}, \quad \frac{\partial}{\partial v}\boldsymbol{r}_\lambda = \lambda^{-1}\mathrm{Ad}(\Phi^{-1})\underline{e}_2, \tag{4.9}$$

$$\frac{\partial}{\partial u}\boldsymbol{n}_\lambda = \lambda\mathrm{Ad}(\Phi^{-1})\{\cosh\omega\,\underline{e}_2 + \sinh\omega\,\underline{e}_3\}, \quad \frac{\partial}{\partial v}\boldsymbol{n}_\lambda = \lambda^{-1}\mathrm{Ad}(\Phi^{-1})\underline{e}_3. \tag{4.10}$$

From these formulas, we get the required result. □

Note that Eq. (4.6) is a formula of Sym's type [24]. If we take

$$\boldsymbol{e}_1 := \mathrm{Ad}(\Phi^{-1})\left\{-\frac{\mathrm{e}^\omega}{2}(\underline{e}_2 + \underline{e}_3)\right\}, \qquad \boldsymbol{e}_2 := \mathrm{Ad}(\Phi^{-1})\left\{\frac{\mathrm{e}^\omega}{2}(\underline{e}_2 - \underline{e}_3)\right\} \tag{4.11}$$

and $\lambda = 1$, we obtain (1.2) with Eqs. (1.5) and (1.6), i.e. $\boldsymbol{r} := \boldsymbol{r}_1$ is just the surface considered in Section 2 and (u, v) are asymptotic Chebyshev coordinates.

Let $u_1 = \lambda u$, $v_1 = v/\lambda$, the fundamental forms of $\boldsymbol{r}_\lambda$ can be written as

$$\mathrm{I}_\lambda = -\mathrm{d}u_1^2 - 2\sinh\omega\left(\frac{u_1}{\lambda}, \lambda v_1\right)\mathrm{d}u_1\,\mathrm{d}v_1 + \mathrm{d}v_1^2. \tag{4.12}$$

$$\mathrm{II}_\lambda = -2\cosh\omega\left(\frac{u_1}{\lambda}, \lambda v_1\right)\mathrm{d}u_1\,\mathrm{d}v_1. \tag{4.13}$$

Thus, $\omega^\lambda(u_1, v_1) := \omega(\lambda^{-1}u_1, \lambda v_1)$ is a solution to

$$\omega^\lambda_{u_1 v_1} = \cosh\omega^\lambda, \tag{4.14}$$

and hence, $\boldsymbol{r}_\lambda$ is a surface corresponding to ω^λ in the sense of Theorem 1.1. In other words, $\boldsymbol{r}_\lambda$ is the Lie transformation of $\boldsymbol{r}_1$ [7]. Thus, we have

Theorem 4.2. *The loop of time-like surfaces $\boldsymbol{r}_\lambda$ with $K = 1$ and two imaginary principal curvatures are the Lie transformations of $\boldsymbol{r}_1$.*

Similar results for other kinds of surfaces of constant curvature have been obtained in [11].

Remark 4.3. The quadruple $\{\boldsymbol{r}_\lambda, \mathrm{Ad}(\Phi^{-1})\underline{e}_2, \mathrm{Ad}(\Phi^{-1})\underline{e}_3, \boldsymbol{n}_\lambda\}$ constitutes a system of moving orthonormal frames on $\boldsymbol{r}_\lambda$. This is the geometrical meaning of the Lax pair (3.1). The

Darboux transform Φ^1 induces a new loop $\{\boldsymbol{r}_\lambda^1\}$ of time-like surfaces of constant curvature $K = 1$ with two imaginary principal curvatures. We call each $\boldsymbol{r}_\lambda^1$ the Darboux transform of each $\boldsymbol{r}_\lambda$.

Theorem 4.4. *Let* $\boldsymbol{r} : S \to \boldsymbol{R}^{2,1}$ *be a time-like surface of constant curvature* $K = 1$ *with two imaginary principal curvatures. Let* Φ *be a solution to the Lax equation* (3.1) *with* $\det\Phi = 1$. *Take a pure-imaginary number* $\lambda_0 \neq 0$ *and Darboux matrix* $\mathcal{D}(\lambda)$ *determined by* λ_0. *Put* $\tilde{\Phi}^1 := \Phi^1/\sqrt{\det\mathcal{D}}$. *Then the Darboux transform* $\boldsymbol{r}_\lambda^1$ *of* $\boldsymbol{r}_\lambda$ *is*

$$\boldsymbol{r}_\lambda^1 = 2(\tilde{\Phi}^1)^{-1}\frac{\partial}{\partial t}\tilde{\Phi}^1, \quad \lambda = \pm e^{-t}, \quad t \in \boldsymbol{R}. \tag{4.15}$$

The pseudo-spherical line congruence correponding to $\mathcal{D}$ *is given by the following formula*

$$\begin{aligned} \boldsymbol{r}_\lambda^1 &= \boldsymbol{r}_\lambda + 2\mathrm{Ad}(\Phi^{-1})\left\{\left(\frac{\mathcal{D}}{\sqrt{\det\mathcal{D}}}\right)^{-1}\frac{\partial}{\partial t}\left(\frac{\mathcal{D}}{\sqrt{\det\mathcal{D}}}\right)\right\} \\ &= \boldsymbol{r}_\lambda + 2\mathrm{Ad}(\Phi^{-1})\left\{\frac{\pm\lambda^2}{\lambda^2-\lambda_0^2}\underline{1} + \mathcal{D}^{-1}\frac{\partial}{\partial t}\mathcal{D}\right\}. \end{aligned} \tag{4.16}$$

Here $\underline{1}$ *denotes the identity matrix.*

Proof. First, we notice that $\det\mathcal{D}(\lambda) = 1 - (\lambda^2/\lambda_0^2) > 0$, since, λ_0 is pure-imaginary. The normalized matrix $\tilde{\Phi}^1$ is a solution to (3.1) with $\det\tilde{\Phi}^1 = 1$. Thus, by Proposition 4.1, the Darboux transform $\boldsymbol{r}_\lambda^1$ is given by Eq. (4.12). Since $\mathcal{D}/\sqrt{\det\mathcal{D}} \in SL_2\boldsymbol{R}$, $(\mathcal{D}/\sqrt{\det\mathcal{D}})^{-1}(\partial/\partial t)(\mathcal{D}/\sqrt{\det\mathcal{D}}) \in \underline{g}$. Thus, Eq. (4.13) gives the pseudo-spherical line congruence corresponding to $\mathcal{D}$. □

Thus, if we know a time-like surface of constant curvature $K = 1$ with imaginary principal curvatures and the solution Φ of Lax pair, then an infinite series of surfaces of the same characters together with the solutions of Lax pair can be obtained successively by Eqs. (2.1), (3.6) and (4.15). The algorithm consists of *elementary operations* (algebraic operation and substitution) only, if we note that, in the algorithm, we need only $e^{\omega'}$ rather than ω'. However, the seed solution with explicit expressions is to be found.

The above discussions is mainly of local character. It is interesting to develope those results to a global theory.

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Appendix

A complete list of surfaces of constant Gaussian curvature and pseudo-spherical congruences in $\boldsymbol{R}^3$ and $\boldsymbol{R}^{2,1}$

Space	Surface	Congruence and BT	DT	Equation
$\boldsymbol{R}^3$	$\boldsymbol{r}$			
	$\boldsymbol{r}'$	$\boldsymbol{r}' = \boldsymbol{r} + l(\cos(\alpha'/2)\boldsymbol{e}_1 + \sin(\alpha'/2)\boldsymbol{e}_2)$	α	$\alpha_{uu} - \alpha_{vv} = \sin\alpha$
	$K = -1$	$\boldsymbol{e}_1^2 = \boldsymbol{e}_2^2 = 1, \boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = 0$	$\downarrow\uparrow$	
	$\mathrm{I} = \cos^2(\alpha/2)\mathrm{d}u^2 + \sin^2(\alpha/2)\mathrm{d}v^2$		α'	$\alpha'_{uu} - \alpha'_{vv} = \sin\alpha'$
	$\mathrm{II} = \cos(\alpha/2)\sin(\alpha/2)(\mathrm{d}u^2 - \mathrm{d}v^2)$			
$\boldsymbol{R}^{2,1}$	$\boldsymbol{r}$			
	$\boldsymbol{r}'$	$\boldsymbol{r}' = \boldsymbol{r} + l(\cos(\alpha'/2)\boldsymbol{e}_1 + \sin(\alpha'/2)\boldsymbol{e}_2)$	α	$\alpha_{uu} - \alpha_{vv} = -\sin\alpha$
	$K = 1$ space-like	Space-like congruence	$\downarrow\uparrow$	
	$\mathrm{I} = \cos^2(\alpha/2)\mathrm{d}u^2 + \sin^2(\alpha/2)\mathrm{d}v^2$	$\boldsymbol{e}_1^2 = \boldsymbol{e}_2^2 = 1, \boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = 0$	α'	$\alpha'_{uu} - \alpha'_{vv} = -\sin\alpha'$
	$\mathrm{II} = -\cos(\alpha/2)\sin(\alpha/2)(\mathrm{d}u^2 - \mathrm{d}v^2)$			
$\boldsymbol{R}^{2,1}$	$\boldsymbol{r}$			
	$\boldsymbol{r}'$	$\boldsymbol{r}' = \boldsymbol{r} + l(\sinh(\alpha'/2)\,\boldsymbol{e}_1 + \cosh(\alpha'/2)\,\boldsymbol{e}_2)$	α	$\alpha_{uu} - \alpha_{vv} = \sinh\alpha$
	$K = 1$ time-like	Time-like congruence	$\downarrow\uparrow$	
	Principal curvatures $\kappa_1 \neq \kappa_2$, real	$\boldsymbol{r}' = \boldsymbol{r} + l(\cosh(\alpha'/2)\,\boldsymbol{e}_1 + \sinh(\alpha'/2)\,\boldsymbol{e}_2)$	α'	$\alpha'_{uu} - \alpha'_{vv} = \sinh\alpha'$
	$\mathrm{I} = \cosh^2(\alpha/2)\mathrm{d}u^2 - \sinh^2(\alpha/2)\mathrm{d}v^2$	Space-like congruence		
	or $\sinh^2(\alpha/2)\mathrm{d}u^2 - \cosh^2(\alpha/2)\mathrm{d}v^2$	$\boldsymbol{e}_1^2 = -\boldsymbol{e}_2^2 = 1, \boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = 0$		
	$\mathrm{II} = \cosh(\alpha/2)\sinh(\alpha/2)(\mathrm{d}u^2 - \mathrm{d}v^2)$			
$\boldsymbol{R}^{2,1}$	$\boldsymbol{r}$	$\boldsymbol{r}' = \boldsymbol{r} + l(a\boldsymbol{e}_1 + b\boldsymbol{e}_2)$	ω	$\omega_{uv} = (1/2)\mathrm{e}^{\omega}$
	$\boldsymbol{r}'$	$ab\,\mathrm{e}^{\omega} = 1$	$\downarrow\uparrow$	
	$K = 1$ time-like	Space-like congruence	ω'	$\omega'_{uv} = -(1/2)\mathrm{e}^{\omega'}$
	$\kappa_1 = \kappa_2$	$ab\,\mathrm{e}^{\omega} = -1$		
	$\mathrm{I} = -\mathrm{e}^{\omega}\,\mathrm{d}u\,\mathrm{d}v + \mathrm{d}v^2$	Time-like congruence		
	$\mathrm{II} = -\mathrm{e}^{\omega}\,\mathrm{d}u\,\mathrm{d}v$	$\boldsymbol{e}_1^2 = \boldsymbol{e}_2^2 = 0, \boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = \mathrm{e}^{\omega}/2$		

Appendix. (*Continued*)

Space	Surface	Congruence and BT	DT	Equation
$\boldsymbol{R}^{2,1}$	$\boldsymbol{r}$	$\boldsymbol{r}' = \boldsymbol{r} + l(a\,\boldsymbol{e}_1 + b\,\boldsymbol{e}_2)$		
	$\boldsymbol{r}'$	$ab\,\mathrm{e}^{\omega} = 1$	ω	$\omega_{uv} = \cosh\omega$
	$K = 1$ time-like	Space-like congruence	$\downarrow\uparrow$	
	κ_1, κ_2 imaginary	$ab\,\mathrm{e}^{\omega} = -1$	ω'	$\omega'_{uv} = \cosh\omega'$
	$\mathrm{I} = -\mathrm{d}u^2 - 2\sinh\omega\,\mathrm{d}u\,\mathrm{d}v + \mathrm{d}v^2$	Time-like congruence		
	$\mathrm{II} = -2\cosh\omega\,\mathrm{d}u\,\mathrm{d}v$	$\boldsymbol{e}_1^2 = \boldsymbol{e}_2^2 = 0$, $\boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = \mathrm{e}^{\omega}/2$		
$\boldsymbol{R}^{2,1}$	$\boldsymbol{r}$			
	$K = -1$ time-like			
	$\mathrm{I} = \cos^2(\alpha/2)\mathrm{d}u^2 - \sin^2(\alpha/2)\mathrm{d}v^2$	$\boldsymbol{r}' = \boldsymbol{r} + l(\cosh(\alpha'/2)\,\boldsymbol{e}_1 + \sinh(\alpha'/2)\,\boldsymbol{e}_2)$	α	$\Delta\alpha = \sin\alpha$
	$\mathrm{II} = \cos(\alpha/2)\sin(\alpha/2)(\mathrm{d}u^2 - \mathrm{d}v^2)$	$\boldsymbol{e}_1^2 = -\boldsymbol{e}_2^2 = 1$, $\boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = 0$	$\downarrow\uparrow$	
	$\boldsymbol{r}'$	$\boldsymbol{r} = \boldsymbol{r}' + l(\cos(\alpha/2)\boldsymbol{e}'_1 + \sin(\alpha/2\boldsymbol{e}'_2))$	α'	$\Delta\alpha' = \sinh\alpha'$
	$K = -1$ space-like	Space-like congruence		
	$\mathrm{I} = \cosh^2(\alpha'/2)\mathrm{d}u^2 + \sinh^2(\alpha/2)\mathrm{d}v^2$	$\boldsymbol{e}_1'^2 = \boldsymbol{e}_2'^2 = 1$, $\boldsymbol{e}'_2 \cdot \boldsymbol{e}'_2 = 0$		
	$\mathrm{II} = \cosh(\alpha'/2)\sinh(\alpha'/2)(\mathrm{d}u^2 + \mathrm{d}v^2)$			

References

[1] A.V. Bäcklund, Einiges über Curve und Flächentransformationen, Lund Universitets Arsskrift, Vol. 10, 1885.

[2] B.M. Barbashov, V.V. Nesterenko, Differential geometry and nonlinear field models, Fortschritte Phys. 28 (1980) 427–464.

[3] B.M. Barbashov, V.V. Nesterenko, A.M. Chervjakov, Generalization of the relativistic string model in the scope of the geometric approach (in Russian), Theret. Mat Fiz. 45 (1980), 365–376, English translation: Theoret. and Math. Phys. 45 (1980), 1082–1089.

[4] S.S. Chern, Geometrical interpretation of sinh-Gordon equation, Ann. Polon. Math. 39 (1980) 74–80.

[5] G. Darboux, Leçons sur la théorie des surfaces, Vol. 2, 2nd Edition, Gauthier-Villars, Paris, 1915.

[6] J. Dorfmeister, J.-I. Inoguchi, M. Toda, Weierstraß type representation of timelike surfaces with constant mean curvature, preprint, 2001.

[7] L.P. Eisenhart, A Treatise on the Differential Geometry on Curves and Surfaces, Ginn and Company, 1909, reprinted by Dover, 1960.

[8] C.H. Gu, On the motion of a string in curved spacetime, in: N. Hu (Ed.), Proceedings of the third Marcel Grossmann Meeting on General Relativity (1982, Shanghai), Science Press and North-Holland Publishing, 1983, pp. 139–142.

[9] C.H. Gu, On the harmonic maps from $\boldsymbol{R}^{1,1}$ to $S^{1,1}$, J. Reine Angew. Math. 346 (1984) 101–109.

[10] C.H. Gu (Ed.), Soliton Theory and its Applications, Springer-Verlag and Zhejiang Science and Technology Publishing House, 1995.

[11] C.H. Gu, H.S. Hu, Z.X. Zhou, Darboux Transformations in Solition Theory and its Geometric Applications (in Chinese), Modern Mathematical Series, Shanghai Sci. Tech. Publ., 1999.

[12] H.S. Hu, The geometry of sine-Laplace, sinh-Laplace equations, in: N. Hu (Ed.), Proceedings of the 3rd Marcel Grossman Meeting on General Relativity (1982, Shanghai), Science Press and North-Holland Publishing, 1983, pp. 1073–1076.

[13] H.S. Hu, The construction of hyperbolic surfaces in three-dimensional Minkowski space and sinh-Laplace equations, Acta Math. Sinica 1 (1985) 79–86.

[14] H.S. Hu, On the geometry of sinh-Gordon equation, in: Beirao da Veiga, et al. (Eds.), Qualitative Aspects and Applications of Nonlinear Evolution Equations (Trieste, 1993) World Sci. Publishing, River Edge, NJ, 1994, pp. 35–47.

[15] H.S. Hu, Darboux transformations between $\Delta\alpha = \sinh\alpha$ and $\Delta\alpha = \sin\alpha$ and the application to pseudo-spherical congruence in $\boldsymbol{R}^{2,1}$, Letts. Math. Phys. 48 (1999) 187–195.

[16] Y.Z. Huang, Bäcklund theorem in three-dimensional Minkowski space and their higher-dimensional generalization (in Chinese), Acta Math. Sinica 29 (1986) 684–690.

[17] J.-I. Inoguchi, Timelike surfaces of constant mean curvature in Minkowski 3-surfaces, Tokyo J. Math. 21 (1998) 141–152.

[18] J.-I. Inoguchi, Darboux transformations on timelike constant mean curvature surfaces, J. Geom. Phys. 32 (1999) 57–78.

[19] V.B. Matveev, M.A. Salle, Darboux Transformations and Solitons, Springer Series in Nonlinear Dynamics, Springer, Heidelberg, 1991.

[20] L. McNertney, One-parameter families of surfaces with constant curvature in Lorentz three-space, Ph.D. Thesis, Brown University, 1980.

[21] T.K. Milnor, Harmonic maps and classical surface theory in Minkowski three-space, Trans. Am. Math. Soc. 280 (1983) 161–185.

[22] B. Palmer, Bäcklund transformation for surfaces in Minkowski space, J. Math. Phys. 31 (1990) 2872–2875.

[23] C. Rogers, W.R. Shadwick, Bäcklund Transformations and their Applications, Academic Press, New York, 1982.

[24] A. Sym, Soliton surfaces and their applications–soliton geometry from spectral problems, in: R. Martini (Ed.), Geometric Aspects of the Einstein Equations and Integrable Systems, Lecture Notes in Phys. 239 (1985), Springer, Berlin, pp. 154–231.

[25] C. Tian, Bäcklund transformations on surface with $K = -1$ in $R^{2,1}$, J. Geom. Phys. 22 (1997) 212–218.

[26] D. Wójcik, J. Cieśliński (Eds.), Nonlinearity and Geometry, Polish Scientific Publishers PWN, Warszawa, 1988.